STABILITY OF A ROTATING LIQUID HEATED FROM BELOW WITH RESPECT TO PERIODIC PERTURBATIONS

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The coupling of a system of ordinary equations for convection [1] leads to the conclusion that a perturbation occurring due to the heating of a liquid from below always varies monotonically with time [2]. Presence of magnetic field (in a conducting liquid) or of a rotation render the equations of motion uncoupled and therefore the stability of an initially balanced system and the nature of perturbations causing its collapse should be investigated separately. Chandrasekhar has considered the effect of magnetic field [3] and of rotation [4,5] on the convection in a horizontal plane layer. The consideration of a layer of infinite length which ordinarily admits the analysis of a phenomenon in a "pure form" in this particular case would only obscure the physical nature of the problem. In a finite strip, however, (all dimensions of which are of the same order) these basically new effects which are occasioned by the magnetic field or by rotation, are discernible in a distinct fashion.

Below, using a simple example, one considers the effect of rotation on the stability of a liquid heated from below which occupies a closed space whose linear dimensions are of the same order in all directions. As will be shown in Section 6, the magnetic field in a conducting liquid is equivalent to the rotation of liquid with regard to all aspects concerning its stability.

1. In a gravitational field

$$\mathbf{g} = -g\boldsymbol{\gamma}, \qquad \boldsymbol{\gamma}^2 = 1 \tag{1.1}$$

the liquid occupies a volume, the walls of which move with a uniform angular velocity

$$\mathbf{\Omega} = \Omega\boldsymbol{\gamma} \tag{1.2}$$

rotating around a common axis. Then in the basic stationary motion of the liquid is a solid body rotation. The rotation of a solid body is always stable [6]. If one heats the liquid from below, establishing thus a steady vertical temperature gradient

$$\nabla T_0 = -A\boldsymbol{\gamma} \tag{1.3}$$

then, due to the continuity, the basic motion will be stable for sufficiently small values of A. One needs to add to the ordinary convection equations [1] the terms for centrifugal and Coriolis forces (we shall consider a slow motion developing within the liquid) and retain only terms which are linear with regard to perturbation. In the following it is assumed that the following condition is satisfied*

$$\Omega^2 l \ll g \quad (l \text{ is the characteristic dimension of the considered layer}) \tag{1.4}$$

Then the equations of motion and of thermal conduction are

$$\begin{aligned} \dot{\mathbf{v}} &= -\nabla f + \nu\nabla^2\mathbf{v} + \alpha g\boldsymbol{\gamma}T + 2\Omega(\mathbf{v}\times\boldsymbol{\gamma}) \\ \dot{T} &= A\boldsymbol{\gamma}\mathbf{v} + \chi\nabla^2 T, \qquad \nabla\mathbf{v} = 0 \end{aligned} \tag{1.5}$$

We introduce characteristic parameters l, v_1, T_1, where v_1 and T_1 are determined in such a way that

$$(v_1 / T_1)^2 = \alpha g\chi / A\nu \tag{1.6}$$

Then Equations (1.5) become

$$\begin{aligned} \dot{\mathbf{v}} &= -\nabla f + \nabla^2\mathbf{v} + C\boldsymbol{\gamma}T + D(\mathbf{v}\times\boldsymbol{\gamma}) \\ P\dot{T} &= C\boldsymbol{\gamma}\mathbf{v} + \nabla^2 T, \qquad \nabla\mathbf{v} = 0 \end{aligned} \tag{1.7}$$

The following dimensionless quantities enter into these equations

$$P = \nu / \chi, \qquad C^2 = \alpha g A l^4 / \nu\chi, \qquad D^2 = 4\Omega^2 l^4 / \nu^2$$

Here, P is the Prandtl number, C^2 is the Rayleigh number, D^2 is the Taylor number.

The linear equations (1.7) do not contain time explicitly. Consequently, all the quantities may be considered to be proportional to $\exp(\lambda t)$

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* That is, one does not consider here fast movement of large volumes.

and we should investigate the limiting case

$$\begin{aligned} \lambda \mathbf{v} &= -\nabla f + \nabla^2 \mathbf{v} + C\gamma T + D(\mathbf{v} \times \gamma) \\ \lambda P T &= C\gamma \mathbf{v} + \nabla^2 T, \qquad \nabla \mathbf{v} = 0 \end{aligned} \tag{1.8}$$

with corresponding boundary conditions on the walls of the layer. The sign of the real part of λ determines the stability: the equilibrium is stable when $\operatorname{Re} \lambda < 0$.

2. As an example we shall consider a cubic volume of unit side length. The following will be the boundary conditions on the walls of cube

$$\begin{aligned} T = v_z = \frac{\partial^2 v_z}{\partial z^2} = 0 \qquad & \text{on the upper and lower wall} \\ v_n = \frac{\partial v_z}{\partial n} = \frac{\partial T}{\partial n} = 0 \qquad & \text{on the side walls} \end{aligned} \tag{2.1}$$

Here $\mathbf{n}$ is a unit vector normal to the wall (the z axis is directed upward.) These conditions are somewhat artificial; however*, the obtained results give qualitatively a correct picture of the behavior of liquid within the container with solid walls. Equations (1.8) with the boundary conditions (2.1) can be solved immediately. For the coordinate projections of velocity and temperature we obtain

$$\begin{aligned} v_x &= v_x^\circ \sin m\pi x \cos n\pi y \cos l\pi z \\ v_y &= v_y^\circ \cos m\pi x \sin n\pi y \cos l\pi z \qquad (m, n, l = 1, 2, \ldots) \\ \{v_z, T\} &= \{v_z^\circ, T^\circ\} \cos m\pi x \cos n\pi y \sin l\pi z \end{aligned} \tag{2.2}$$

The convective motion resulting is characterized by periodicity in all directions. The entire volume of the cube is subdivided into equal cells; in each of them the liquid moves in the same fashion. Each cell has the form of a parallelogram whose sides bear the ratio $m^{-1} : n^{-1} : l^{-1}$ to each other. On the walls of the cells the conditions (2.1) are satisfied.

In order to find the admissible values of λ we proceed as follows. To the first equation of the system (1.8) we apply the rotational operator (curl) twice. We obtain

$$\lambda (\nabla \times \mathbf{v}) = \nabla^2 (\nabla \times \mathbf{v}) + C(\nabla T \times \gamma) + D(\gamma \nabla)\mathbf{v} \tag{2.3}$$

$$\lambda \nabla^2 \mathbf{v} = \nabla^4 \mathbf{v} - C[(\gamma \nabla)\nabla T - \gamma \nabla^2 T] - D(\gamma \nabla)(\nabla \times \mathbf{v}) \tag{2.4}$$

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* Refer to the work of Rayleigh where the theory of stability has been originally formulated.

We substitute solutions (2.2) into the second Equation (1.8) and also into (2.3) and (2.4), dot multiplied by γ. After simple transformations one obtains for the amplitudes of velocities and temperatures a system of three homogeneous algebraic equations. The condition for their solution

$$\begin{vmatrix} (\lambda + k^2)k^2 & -(k^2 - \pi^2 l^2)C & \pi l D \\ C & -(\lambda P + k^2) & 0 \\ \pi l D & 0 & -(\lambda + k^2) \end{vmatrix} = 0 \tag{2.5}$$

$$k^2 \equiv \pi^2[m^2 + n^2 + l^2] \tag{2.6}$$

gives the equation for the eigenvalues λ of the limiting problem (1.8).

3. We shall consider in greater detail the stability of the equilibrium with respect to the largest motion(among all those possible in the cube). To this end we shall put into (2.5) $m = n = l = 1$. We introduce the following notation

$$\mu = \lambda / 3\pi^2, \quad r = 2C^2 / 27\pi^4, \quad \tau = D^2 / 27\pi^4 \tag{3.1}$$

Developing the determinant (2.5) and utilizing (3.1) we obtain for μ the cubic equation

$$\alpha\mu^3 + \beta\mu^2 + \gamma\mu + \delta = 0 \tag{3.2}$$

where

$$\alpha = P, \quad \beta = 2P + 1, \quad \gamma = 2 + P + P\tau - r, \quad \delta = 1 + \tau - r \tag{3.3}$$

The roots of Equation (3.2) coincide with the eigenvalues of the system (1.8) up to a multiplying factor $3\pi^2$. Therefore (see the end of Section 1) the onset of instability is equivalent to the appearance in solutions of Equations (3.2) of a real part in μ equal to zero. Inasmuch as the problem here consists in the appearance of a convective motion following the instability - both, for the stationary and the nonstationary case - then having in mind both of these possibilities it is necessary to look for solutions in which (1) the imaginary part of μ equals zero, that is, solutions which are time independent, and also (2) solutions where the imaginary part of μ differs from zero, that is, solutions which depend periodically on time.

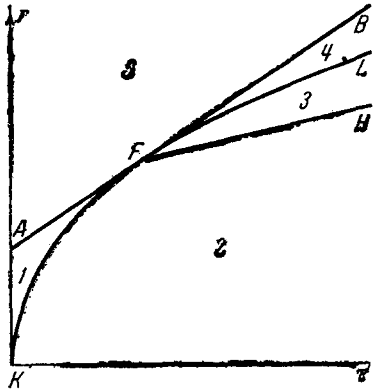
Solutions of the first type are possible when δ (the free member of equation (3.2)) equals zero, i.e.

$$r = 1 + \tau \tag{3.4}$$

Solutions of the second type occur when the coefficients (3.3) fulfill the relation $\alpha\delta = \beta\gamma$ i.e. for

$$r = 2(1+P) + \frac{2P^2}{1+P}\tau \tag{3.5}$$

Two out of three roots of Equation (3.2) are imaginary and conjugate. The square of their modulus equals



$$|\mu|^2 = \frac{1-P}{1+P}\tau - 1 \tag{3.6}$$

From that it can be seen that

$$P < 1 \tag{3.7}$$

is a necessary condition for the existence of periodic solutions (imaginary eigenvalues). Liquid metals satisfy this condition. For example, for mercury at room temperature $P = 1/40$. In the following it will be assumed that condition (3.7) is fulfilled.

The relations $r = r(\tau)$ expressed by Equations (3.4) and (3.5) are shown on Fig. 1 as curves *AB* and *FH*, respectively. On the same figure is drawn the curve *KFL*, the equation of which is

$$\Delta(r, \tau) = 4Pr^3 - [12P^2\tau - (1-P)^2]r^2 + 4P\tau[3P^2\tau - 5(1-P)^2]r - 4\tau[P^2\tau + (1-P)^2]^2 = 0 \tag{3.8}$$

$\Delta(r, \tau)$ is the discriminant of the cubic Equation (3.2). In the area located above the curve *KFL*, Equation (3.2) has only real solutions; underneath that curve two roots of the equation are complex conjugate. The coordinates of point *F* are

$$r_* = 2/(1-P), \qquad \tau_* = (1+P)/(1-P) \tag{3.9}$$

4. The analysis indicates that out of the three roots of Equation (3.2) one (let us denote it by μ_0) is always negative, that is the corresponding forced perturbation decays for arbitrary values of Rayleigh or Taylor numbers.

Table 1 shows the values the remaining two roots μ_1 and μ_2 acquire in each of the five zones defined by means of numbers in Fig. 1. The boundaries between these zones are given by *AB*, *FH* and *KFL*, the equations for which have been given in the previous paragraph.

It is seen from the table that in the first two zones the equilibrium of the liquid is always stable but the damping of perturbations occurs differently in each of them: in domain 1 it is always monotonic and in domain 2 perturbations similar to damped oscillations are possible. In domains 3, 4 and 5 the equilibrium of the liquid is unstable. In domains 4 and 5 the resulting perturbations grow monotonically with time; in domain 4 perturbations of two types (or linear combination thereof) appear to be unsafe, while in domain 5 only one type is unsafe.

TABLE 1.

Zones	$\text{Im}\,\mu_1 \neq 0$	Nonequilibrium	
		$\text{Re}\,\mu_1 > 0$	$\text{Re}\,\mu_2 > 0$
1			
2	×		
3	×	×	×
4		×	×
5		×	

A different situation obtains in domain 3. An arbitrary perturbation can be described here as a linear combination of a single monotonically damped and two oscillatory motions with increasing amplitude (corresponding to the numbers μ_0, μ_1, μ_2). Obviously, no such combination will yield a solution monotonically increasing in time. Therefore, a stationary regime is altogether impossible in domain 3. The equilibrium in this domain is unstable with regard to periodic perturbations whose frequency equals $\pm\,\text{Im}\,3\pi^2\mu_1$.

5. On Fig. 1, in addition to the indicated stability curves for the most intense perturbations with $m = n = l = 1$, it is necessary to plot similar curves for all possible m, n, l.

TABLE 2.

m	n	l	C_*^2	D_*^2	tg φ	$10^3 \cdot$tgψ
1	1	1	2 700	2 800	0.50	0.61
2	1	1	8 600	22 100	0.20	0.24
2	2	1	18 200	74 500	0.12	0.15
1	1	2	21 600	5700	2.00	2.44

Table 2 contains data necessary for the construction of stability curves corresponding to the first four different combinations of numbers m, n, l. This calculation is done for mercury $P = 1/40$. In the fourth and fifth row of Table 2 are given the coordinates of point F of the graph whose abscissae indicate the Taylor number and the ordinates the Rayleigh number. The tangents of the angles between the line AB and the abscissa axis (angle φ) and the line FH and the abscissa axis (angle ψ) are given in the last two rows of the table. It is necessary to remember however, that these numerical estimates are obtained from the solution of a problem with unrealistic boundary conditions (2.1).

The above indicated effects will be observed in an experiment with somewhat higher values of Rayleigh and Taylor numbers.

6. In [8] it was shown that the stability of equilibrium of a conducting liquid placed in a magnetic field

$$\mathbf{H} = H\gamma \tag{6.1}$$

is determined by corresponding numbers λ (in [8] the numbers λ have reverse sign) of a limiting problem

$$\begin{aligned} \lambda \mathbf{v} &= -\nabla f + \nabla^2 \mathbf{v} + C\gamma T + M(\gamma\nabla)\mathbf{h}, \qquad \nabla \mathbf{v} = \nabla \mathbf{h} = 0 \\ \lambda PT &= C\gamma \mathbf{v} + \nabla^2 T, \qquad \lambda N \mathbf{h} = \nabla^2 \mathbf{h} + M(\gamma\nabla)\mathbf{v} \end{aligned} \tag{6.2}$$

Here $\mathbf{h}$ is the additional magnetic field which is generated within the liquid by the outside field H. Beside P and C, the following dimensionless quantities enter into these equations

$$N = 4\pi\nu\sigma / c^2, \qquad M^2 = H^2\sigma l^2 / \rho\nu c^2 \qquad \text{(square of the Hartman number)}$$

On the nonconducting walls of the cube we set the following boundary conditions

$$\begin{aligned} &h_z && \text{is continuous on the upper and lower wall} \\ &\partial h_z / \partial n = 0 && \text{on the side walls} \end{aligned} \tag{6.3}$$

With the boundary conditions (2.1), (6.3) the problem (6.2) can readily be solved. Retaining for μ and r the notations (3.1) and introducing $S = M^2/9\pi^2$, after simple calculations we obtain for μ the Equation (3.2) with coefficients

$$\begin{aligned} \alpha &= PN, \qquad \gamma = 1 + P + N + Ps - Nr \\ \beta &= P + N + PN, \qquad \delta = 1 + s \angle r \end{aligned} \tag{6.4}$$

Instead of (3.6) and (3.7) we will have now correspondingly

$$|\mu|^2 = \frac{1}{N^2}\left(\frac{N-P}{1+P}s - 1\right), \qquad P < N \tag{6.5}$$

Mercury does not meet condition (6.5): $P/N \sim 10^5$. It will be noted that the square of the Hartman number enters into (6.4) in exactly in the same way as the Taylor number into (3.3). Therefore, if τ (see the figure) is identified with s, then all that has been said about stability in Section 4 is also valid in the present case.

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BIBLIOGRAPHY

1. Landau, L.D. and Lifshits, E.M., *Mekhanika sploshnykh sred (Mechanics of Continuous Media)*. GITTL, 1953.

2. Sorokin, V.S., Variatsionnyi metod v teorii konvektsii (Variational method in the theory of convection). *PMM* Vol. 17, No. 1, 1953.

3. Chandrasekhar, S., On the inhibition of convection by a magnetic field. *Phil. Mag.* Vol. 43, p. 501, 1952.

4. Chandrasekhar, S., The instability of a layer of fluid heated below and subject to Coriolis forces. I. *Proc. Roy. Soc.* A, Vol. 217, p. 306, 1953.

5. Chandrasekhar, S. and Elbert, D., The instability of a layer of fluid heated below and subject to Coriolis forces. *Proc. Roy. Soc.* A, Vol. 231, p. 198, 1955.

6. Sorokin, V.S., Nelineinye iavleniia v zamknutykh potokakh vblizi kriticheskogo chisla Reinol'dsa (Nonlinear phenomena in closed flows near critical Reynolds number). *PMM* Vol. 25, No. 2, 1961.

7. Lord Rayleigh, On convection currents in a horizontal layer of fluid, when the higher temperature is on the under side. *Phil. Mag.* Vol. 32, p. 529, 1916.

8. Sorokin, V.S. and Sushkin, I.V., Ustoichivost' ravnovesiia podogrevaemoi snizu provodiashchei zhidkosti v magnitnom pole (Stability of the equilibrium of the heated from below conducting liquid in the presence of magnetic field). *ZhETF* Vol. 38, No. 2, 1960.

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